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1994 J. Phys. A: Math. Gen. 27 5673

(http://iopscience.iop.org/0305-4470/27/16/030)

J. Phys. A: Math. Gen. 27 (1994) 5673–5685. Printed in the UK

Narrow-tube approximation of semiclassical quasi-energies: application to the weakly nonlinear Duffing oscillator

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Received 21 February 1994

Abstract. An exact transformation of semiclassical quantization conditions determining quantal quasi-energies of time-periodic Hamiltonian systems is suggested. For motion on vortex tubes centred at stable periodic orbits the classical quasi-energy of the underlying periodic orbit separates out as a single term which is a lower bound for the quantized quasi-energies. A particular approximation involving the classical Lewis invariant for calculating semiclassical quasi-energies near the centre of a perturbed linear response is discussed in some detail.

1. Introduction

The proper framework for describing T-periodic Hamiltonian systems quantum mechanically is the theory of Floquet or quasi-energy states (QES). Quasi-energy states are special solutions of the time-dependent Schrödinger equation with the property

$$\Psi_\epsilon(x, t+T) = \exp(-\mathrm{i}\epsilon T/\hbar)\Psi_\epsilon(x, t) \tag{1a}$$

and having the form

$$\Psi_\epsilon(x, t) = \exp(-\mathrm{i}\epsilon t/\hbar)\Phi_\epsilon(x, t) \tag{1b}$$

where $\Phi_\epsilon(x, t) = \Phi_\epsilon(x, t+T)$ is a T-periodic function and ϵ is called the *quasi-energy* [1]. The quasi-energy states can be viewed as one-dimensional unitary representations of the symmetry group of discrete time translations by nT, $n = 0, \pm1, \pm2, \ldots$, and the quasi-energies ϵ determine the characters $\exp(-\mathrm{i}\epsilon nT/\hbar)$ of the corresponding representation. Hence, the quasi-energy states play the same role as stationary states do for time-independent Hamiltonians. In particular the functions $\Phi_\epsilon(x, t)$ satisfy a kind of 'time-independent' Schrödinger equation

$$\tilde{H}\Phi_\epsilon(x, t) = \epsilon\Phi_\epsilon(x, t) \tag{2}$$

where $\tilde{H} = [H(t) - \mathrm{i}\hbar\partial/\partial t]$ is a Hermitian operator in the product Hilbert space L^2 of T-periodic functions [1]. In this formalism, many quantum-mechanical theorems for the time-independent Schrödinger equation are applicable to the solutions of (2) like the variational principle, the Hellman–Feynman and the Hypervirial theorem, and also the perturbation theory for the quasi-energy states can be developed in close analogy to the time-independent case [1].

0305-4470/94/165673+13$19.50

Recently Breuer and Holthaus [2] employed the Maslov construction of canonical operators to derive two sets of semiclassical quantization conditions for the quasi-energies and Floquet states of periodically time-dependent systems. Whereas the first set is identical to the usual EBK quantization rules selecting those classical vortex tubes (tori) with quantized actions I_n, the second set fixes the corresponding quasi energies ϵ_n (up to multiples of $\hbar 2\pi/T$). The latter requires the integration of the Poincaré–Cartan form $\omega^1 = p\,\mathrm{d}x - H\,\mathrm{d}t$ along a periodic path of the quantized vortex tubes, which is a non-trivial task and has been done analytically only for two cases, namely the periodically forced [2] and the parametrically excited harmonic oscillator [3].

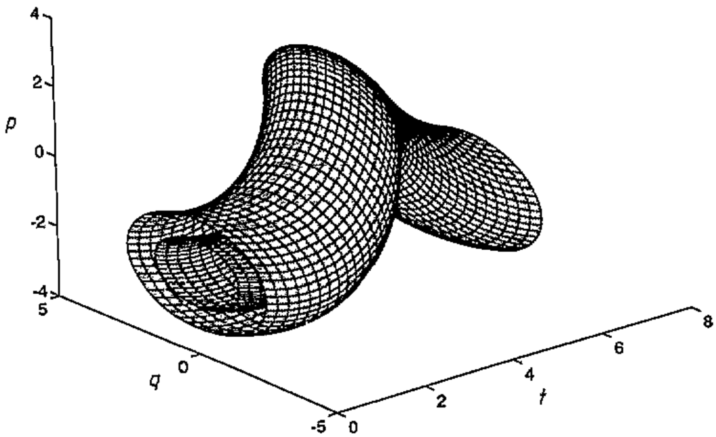

Figure 1. Graphical 3D illustration of a vortex tube spiralling through extended phase space in a helix-like, time-periodic manner.

In order to treat the general case (see figure 1), when the vortex tubes may weave in a complicated (helix-like) manner through extended phase space $\{p, p_t, q, t\}$, the second set of quantization condition has been rewritten by Bensch, Korsch and Mirbach [4] in terms of actual trajectories which wind around the vortex tubes. A first numerical application to a frictionless, periodically driven Duffing oscillator yielded semiclassical quasi-energies in excellent agreement with the exact quantum mechanical ones. However, there are a number of open questions that need further investigations.

In this investigation we focus on an analytic understanding of the general behaviour of quasi-energies near stable periodic motion of the classical system. Typically there are many periodic trajectories, harmonic and subharmonic, in a nonlinear classical oscillator of the Duffing type. When stable, they are centres of elliptic islands of various sizes in the Poincaré cross-section. The nested vortex tubes of the quasi-periodic classical motion near a periodic centre show up as level lines surrounding the fixed point in the Poincaré cross-section, as illustrated in figure 2. Our approach to understanding the quasi-energy spectrum is closely related to this picture of 'local energy wells' in phase space. We develop a narrow-tube approximation for the semiclassical quasi-energies which applies quite generally to any elliptic island of the phase space map. However, we illustrate the details only for the elliptic island corresponding to the main, small-amplitude harmonic response of the periodically forced Duffing oscillator, which is the centre closest to the origin of phase space in figure 2.

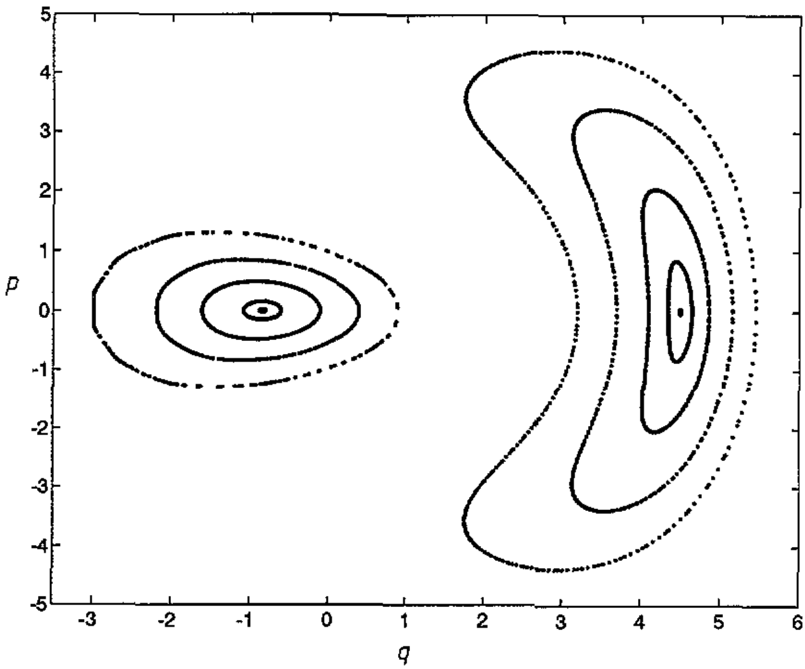

Figure 2. Illustration of the two main stable responses (fixed points in the Poincaré cross-section) for the weakly nonlinear Duffing oscillator ($s = 0.05$). The level lines surrounding the centres correspond to the cross-section of nested vortex tubes. Calculations discussed in the text refer to cases of slightly weaker nonlinearities, where the large-amplitude fixed point is shifted further to the right in the figure.

In section 2 we discuss the Duffing oscillator and a time-dependent canonical transformation to coordinates of the reference frame moving with the periodic response. Semiclassical formulae for determining quasi-energies are also reviewed. Section 3 deals with the centre motion itself and its contribution to the quasi-energy. Approximate, analytic formulae are obtained. In section 4 we develop the narrow-tube approximation for describing the phase space dynamics near the centre motion and from this we obtain quasi-energy formulae. In the adiabatic limit we find simple analytic formulae. Numerical results are discussed in section 5, and some conclusions are given in section 6.

2. The forced Duffing oscillator: perturbed linear case

The time-dependent Hamiltonian

$$H(p, q, t) = \tfrac{1}{2}(p^2 + kq^2 + \tfrac{1}{2}sq^4) - rq\cos(t) \tag{3}$$

describes the classical motion of the excited Duffing oscillator through Hamilton's equations. The Hamiltonian is conveniently written in a form where the linear force constant k, the excitation amplitude r and the strength of the nonlinearity s are the three system parameters. For general system parameters the motion can be either periodic quasi-periodic or irregular (chaotic), depending on initial conditions. For the time being we assume the existence of a stable periodic orbit q_p of period $T = 2\pi$. The motion near the periodic response is correctly described by the nonlinear variational equation, an equation usually written in terms of a relative coordinate $Q = q - q_p$. The correct Hamiltonian corresponding to the relative motion is obtained by a proper time-dependent canonical transformation (see [5] and [6] for a general background):

$$(p, q) \to (P, Q) \equiv (p - \dot{q}_p, q - q_p) \tag{4}$$

where a dot $\dot{}$ denotes a differentiation with respect to time. A time-dependent generating function for the transformation is

$$F_2(P, q, t) = (q - q_p)(P + \dot{q}_p) \tag{5}$$

which, as usual, satisfies the pair of equations

$$p = \frac{\partial F_2}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P}. \tag{6}$$

The transformed Hamiltonian $K(P, Q, t)$ can now be obtained in the standard way [5] from the original one in equation (3):

$$\begin{aligned} K(P, Q, t) &= H(P, Q, t) + (Q + q_p)\ddot{q}_p - P\dot{q}_p - \dot{q}_p^2 - \ddot{q}_p q_p \\ &= \tfrac{1}{2}P^2 + \tfrac{1}{2}(k + 3sq_p^2)Q^2 + sq_p Q^3 + \tfrac{1}{4}sQ^4 \\ &\quad + Q[\ddot{q}_p + kq_p + sq_p^3 - r\cos(t)] \\ &\quad - \{\tfrac{1}{2}\dot{q}_p^2 - \tfrac{1}{2}kq_p^2 - \tfrac{1}{4}sq_p^4 + q_p r\cos(t)\}. \end{aligned} \tag{7}$$

Since we have chosen a periodic solution q_p to the forced Duffing oscillator, the expression multiplying Q in the third line of (7) vanishes and the last line may be identified as the Lagrangian of the centre motion. The simplified expression for the new Hamiltonian is now

$$K(P, Q, t) = \tfrac{1}{2}P^2 + \tfrac{1}{2}(k + 3sq_p^2)Q^2 + sq_p Q^3 + \tfrac{1}{4}sQ^4 - L_p. \tag{8}$$

Apart from the centre motion Lagrangian, L_p, which depends only on the underlying periodic trajectory and does not enter into the dynamical equations of motion, there are other terms in (8) which correspond to a Hamiltonian, h_{rel}, of a parametrically driven, nonlinear (anharmonic) oscillator. The motion of that oscillator are solutions satisfying the differential equation:

$$\ddot{Q} + (k + 3sq_p^2)Q + 3sq_p Q^2 + sQ^3 = 0. \tag{9}$$

This can be seen as a nonlinear variational equation with respect to any periodic solution q_p of the forced Duffing oscillator.

The calculation of quasi-energies can proceed in several ways. The semiclassical procedure [2–4] involves the quantization of vortex tubes in the odd-dimensional extended phase space $\{(P, Q, t)\}$ by means of the Poincaré–Cartan form [5, 7]

$$\omega^1 = P\,\mathrm{d}Q - K\,\mathrm{d}t$$

a geometric generalization of the Lagrangian function. In the present three-dimensional case, two 'geometrically' periodic quantization paths, representing the two different homology classes of the tube, can be constructed such that γ_1 lies on the cross-section of the tube in a Poincaré surface of section $t = 0$, and where γ_2 stretches periodically ($\gamma_2(0) = \gamma_2(T)$) along the tube from $t = 0$ to $t = T$ in such a way that it can be diffeomorphically projected onto the t axis (i.e. onto the interval $[0, T]$). Now, quantizing the first of the integrals, namely

$$\frac{1}{T}\oint_{\gamma_1} \omega^1 = (n + \tfrac{1}{2})\hbar \tag{10}$$

leads to a discrete selection of vortex tubes. The subsequent evaluation of the quasi-energies along these tubes then follows from

$$\epsilon = -\frac{1}{T}\int_{\gamma_2} \omega^1 \tag{11}$$

apart from trivial additional multiples of $\hbar 2\pi/T$; see [2]. Equation (11) can be taken as a definition of the classical quasi-energy of motion on periodic vortex tubes in general, not necessarily the quantized ones.

The integrals $\oint_{\gamma_1} \omega^1$ and $\int_{\gamma_2} \omega^1$ are manifestly invariant under time-dependent, time-periodic canonical transformations [2, 5]. Therefore the two quantization integrals can be expressed in terms of our relative canonical variables P and Q, of the Hamiltonian (8) or some other pair of canonical coordinates P' and Q' with the pertinent Hamiltonian $K'(P', Q', t)$. The invariant 'definition' of the quasi-energy may take a particularly simple form if we choose a path γ_2 with $P' = 0$, which is peculiar to accelerated frames of reference attached to the periodic centre motion q_p. With the restriction $P' = 0$ we are using a path γ_2 which is tracing out one of the turning points of motion along the tube. As an alternative, geometric formula for calculating the classical quasi energy we now have

$$\epsilon = \frac{1}{T}\int_{\gamma_2} K' \, dt. \tag{12}$$

This we think is an interesting classical result. However, equation (12) is not a single-trajectory formula in the ordinary sense and cannot be identified with a time-averaged transformed Hamiltonian. With the Hamiltonian $K(P, Q, t)$ in (8), it is, however, straightforward to convert the L_p part, which is a periodic function of time, to a time-averaged single-trajectory quantity $\langle L_p \rangle_t$. Hence, in this case

$$\epsilon = \frac{1}{T}\int_{\gamma_2} h_{rel} \, dt - \langle L_p \rangle_t. \tag{13}$$

The decomposition of the quasi-energy formula in two parts, as explicitly shown in (13) but also hidden in (12), reveals a quasi-energy structure of local wells at the (periodic) fixed points of the Poincaré map. Classically, the quasi-energy at the stable periodic centre is given by $-\langle L_p \rangle_t$. The nested vortex tubes, as far as they exist, correspond to level lines of increasing quasi-energy. As for developing approximate analytical formulae, the two contributions to the quasi-energy need separate considerations which will be dealt with in section 3 and section 4, respectively.

3. Periodic centre motion

In this section we describe the most dominating harmonic responses of the excited Duffing oscillator when the nonlinearity parameter s is small. We also study in more detail the specific periodic motion q_p which is relevant for the numerical applications in section 5 and for some analytic expressions in section 4 approximating $\langle L_\mathrm{p}\rangle_t$ in the quasi-energy formula (13).

A symmetric, approximate solution with period $1T$ can be sought with a leading term of a general Fourier series given by

$$q_\mathrm{p} = F\cos(t) + \cdots. \tag{14}$$

Inserting and equating the leading-order terms (see [8] for the harmonic balance method), we find

$$(-F + kF + \tfrac{3}{4}sF^3)\cos(t) = r\cos(t). \tag{15}$$

The cubic equation (15) for F suggests the co-existence of three real, periodic solutions that we will briefly describe. Assuming the forcing amplitude r has a fix value, we obtain from (15) the following relation between k and F:

$$k = 1 + \frac{r}{F} - \frac{3}{4}sF^2. \tag{16}$$

There is a negative-F branch of the 'function' $k(F)$ in (16), which corresponds to two out-of-phase responses (for given negative k there are two matching negative Fs), and a positive-F branch corresponding to a single in-phase response. Folding the negative-F branch onto the positive-F side, by taking the modulus of F, we obtain the nonlinear resonance curve in figure 3. The resonance curve for the present Duffing oscillator is tilted in a characteristic way so that large-amplitude responses behave quite differently from responses in a linear oscillator. We see in figure 3 that for given negative and small positive values of k there is now one small-amplitude response and two large-amplitude responses. We know from standard perturbation analysis [9] that the largest solution, which is *in phase* with the excitation, is the stable one together with the small-amplitude out-of-phase solution. The other large solution is unstable.

We are particularly interested in cases where there are several periodic responses, and, hence, competing sets of vortex-tube motion. In the numerical study of section 5 we consider the case $k = 0.168^2 \approx 0.38$, which for sufficiently small forcing r and nonlinearity strength s admits more than one dominating, periodic response of the Duffing oscillator. On the other hand we did not want the appearance of too many periodic (subharmonic) responses or irregular motion of small amplitude in this study, so our choice of k had to meet a requirement of not being too close to zero. Furthermore, the most favourable solution to approximate by the single-term Fourier ansatz (14) is the small-amplitude response. This response therefore specifies the centre motion q_p for the vortex tubes we should like to quantized.

A crude analytic approximation of the small-amplitude solution may be obtained by first neglecting the quadratic term on the right-hand side of equation (16). For $k \neq 1$, we find

$$F = -\frac{r}{1-k} \tag{17}$$

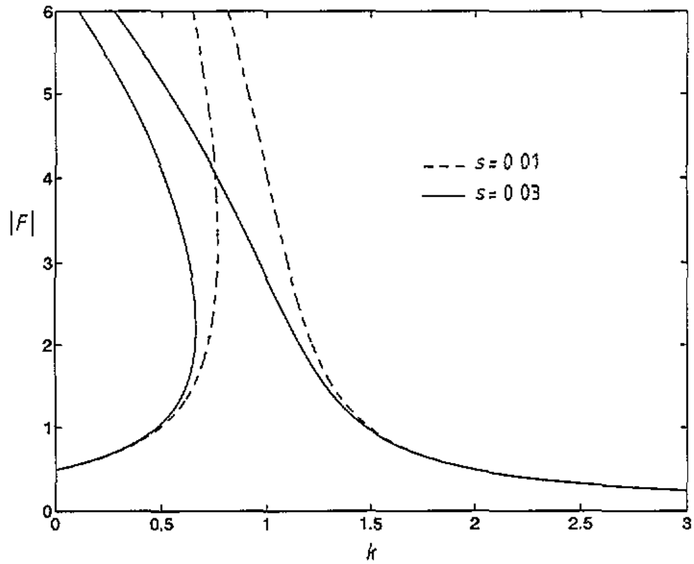

Figure 3. The classical amplitude-response curve from equation (16), for the main harmonic responses of the weakly nonlinear Duffing oscillator. Both curves correspond to an excitation amplitude $r = 0.5$.

which is a fair single-frequency approximation for small excitation amplitudes and k not too close to 1.

It is desirable, however, to find a better approximation for F, which also incorporates the effects of the so far neglected nonlinear term. We suggest rewriting equation (16) in the alternative form:

$$F = \frac{r}{k - 1 + \frac{3}{4}sF^2} \tag{18}$$

and solving by iterating twice, starting with $s = 0$. We then find a second approximation for the amplitude F of the form

$$F = \frac{r}{k - 1 + \frac{3}{4}s\frac{r^2}{(1-k)^2}}. \tag{19}$$

The quasi-energy contribution from the centre motion q_p is given by $\langle -L_p \rangle_t$. This quantity will be calculated both 'exactly' (the notation $\langle -L_p \rangle_{NT}$ referring to the 'exact' narrow-tube result) according to

$$\langle -L_p \rangle_{NT} = \langle -\tfrac{1}{2}\dot{q}_p^2 + \tfrac{1}{2}kq_p^2 + \tfrac{1}{4}sq_p^4 + q_p r\cos(t) \rangle_t \tag{20a}$$

and approximately, according to the approximation $q_p \approx F\cos(t)$ together with (19). Hence, it is straightforward analysis to derive the following approximate centre-motion contribution (denoted $\langle -L_p \rangle_A$):

$$\langle -L_p \rangle_A = \tfrac{1}{4}F^2(1-k) - \tfrac{9}{32}sF^4. \tag{20b}$$

We note that F in (20*b*) actually depends on the nonlinearity parameter s through equation (19).

Next we investigate the contribution from the assumed narrow tubes centred at the periodic response. They are originally described by relative coordinates with the origin moving with the periodic response, as explained earlier.

4. Narrow-tube approximation

We may start from the Hamiltonian in (8) and try to obtain some approximate analytic understanding of the vortex tubes close to the centre. Even if we neglect cubic and quartic terms in the Hamiltonian, the resulting Hill Hamiltonian is not easy to approach. We shall see that it is possible to invoke a further canonical transformation in order to obtain analytic results. In this section we invoke the classical Lewis invariant [10] to separate time and phase space. This approach allows an adiabatic simplification which is also discussed in some detail.

4.1. Lewis invariant approach

In the zeroth-order approximation we consider the quadratic Hamiltonian:

$$K_0(P,Q,t) = \tfrac{1}{2}P^2 + \tfrac{1}{2}(k + 3sq_{\rm p}^2)Q^2 - L_{\rm p}. \tag{21}$$

This is our Hill oscillator that can be treated exactly by a further time-dependent, canonical transformation. Let us introduce the canonical coordinates

$$P' = P\rho(t) - Q\dot{\rho}(t) \qquad Q' = Q/\rho(t) \tag{22}$$

with $\rho(t)$ being an arbitrary positive function, yet to be determined. A time-dependent generating function $G_2(P',Q,t)$ for the canonical transformation equation (22) is given by

$$G_2(P',Q,t) = \frac{1}{\rho(t)}P'Q + \frac{\dot{\rho}(t)}{2\rho(t)}Q^2 \tag{23}$$

satisfying

$$\frac{\partial G_2}{\partial Q} = P \qquad \frac{\partial G_2}{\partial P'} = Q'. \tag{24}$$

The time-dependent canonical transformation of the Hamiltonian K_0 gives

$$\begin{aligned} K_0'(P',Q',t) &= K_0(P(P',Q',t),Q(Q',t),t) + \frac{\partial G_2}{\partial t} \\ &= \frac{1}{2\rho^2(t)}P'^2 + \tfrac{1}{2}[\omega^2\rho^2 + \ddot{\rho}\rho]Q'^2 - L_{\rm p}(t). \end{aligned} \tag{25}$$

We now require $\rho(t)$ to be a periodic solution of the auxiliary Milne equation

$$\ddot{\rho}\rho + \omega^2(t)\rho^2 = \frac{1}{\rho^2} \tag{26}$$

where

$$\omega^2(t) = k + 3sq_{\rm p}^2(t). \tag{27}$$

Then by virtue of such a solution $\rho(t)$, the transformed Hamiltonian (25) has the form

$$K_0' = \frac{1}{2\rho^2(t)}[P'^2 + Q'^2] - L_{\rm p}(t) \tag{28}$$

and K_0' is a periodic function of time because $\rho(t)$ and $L_p(t)$ are so. We have made a transformation to new canonical coordinates in which the so-called Lewis invariant [10], I, takes the form

$$I = \tfrac{1}{2}[P'^2 + Q'^2]. \tag{29}$$

One can verify that the EBK quantization condition for the tubes leads to

$$\frac{1}{2\pi}\oint_{\gamma_1} P'\,\mathrm{d}Q' = I = (n+\tfrac{1}{2})\hbar \tag{30}$$

using the new coordinates and the Lewis invariant. The zeroth-order quasi-energy is now

$$\epsilon_0 = \frac{1}{T}\int_0^T K_0'(t)\,\mathrm{d}t = \langle\rho^{-2}(t)\rangle_t(n+\tfrac{1}{2})\hbar - \langle L_p(t)\rangle_t \tag{31}$$

where we have introduced the time-averaged quantities $\langle\rho^{-2}(t)\rangle_t$ and $\langle L_p(t)\rangle_t$. This is an 'exact' semiclassical result in the narrow-tube limit. In order to get an estimate of the quasi-energy corresponding to the original Hamiltonian function $K(P, Q, t)$, containing cubic and quartic terms in Q, we use the narrow-tube coordinates P' and Q'. The same canonical transformation leads to the Hamiltonian

$$\begin{aligned} K'(P', Q', t) &= K(P, Q, t) + \frac{\partial G_2}{\partial t} - L_p(t) \\ &= \frac{1}{2\rho^2(t)}[P'^2 + Q'^2] + sq_p(t)\rho^3(t)Q'^3 + \tfrac{1}{4}s\rho^4(t)Q'^4 - L_p(t). \end{aligned} \tag{32}$$

This Hamiltonian still depends on both t and (P', Q'). The contour lines I = constant of equation (29) define circles in the (P', Q') coordinates. Going over to polar coordinates (I, φ), i.e. the action-angle variables of K_0, defined by

$$Q' = \sqrt{2I}\cos\varphi \qquad P' = \sqrt{2I}\sin\varphi$$

the transformed full Hamiltonian may be rewritten as

$$K'(I, \varphi, t) = \rho^{-2}(t)I + sq_p(t)\rho^3(t)(2I)^{3/2}\cos^3\varphi + s\rho^4(t)I^2\cos^4\varphi - L_p(t). \tag{33}$$

We now introduce the approximate Hamiltonian $\bar{K}'(I, t)$, which we obtain from $K'(I, \varphi, t)$ in (33) by a time-independent averaging with respect to the angle φ. Hence,

$$\bar{K}'(I, t) = \langle K'(I, \varphi, t)\rangle_\varphi = I\rho^{-2}(t) + \tfrac{3}{8}sI^2\rho^4(t) - L_p(t). \tag{34}$$

The narrow-tube approximation (NT) of the geometric path formula (13) for the quasi-energy now yields

$$\begin{aligned} \epsilon_{\mathrm{NT}} &\equiv \frac{1}{T}\int_0^T \bar{K}'(I, t)\,\mathrm{d}t \\ &= \langle\rho^{-2}\rangle_t(n+\tfrac{1}{2})\hbar + \tfrac{3}{8}s\langle\rho^4(t)\rangle_t(n+\tfrac{1}{2})^2\hbar^2 + \langle -L_p\rangle_{\mathrm{NT}} \end{aligned} \tag{35}$$

where the Lewis invariant has been quantized according to equation (30), and $\langle -L_p\rangle_{\mathrm{NT}}$ has been introduced from (20*a*).

4.2. Adiabatic limit

Approximating the periodic Milne solution by its adiabatic expression $\rho(t) \approx 1/\sqrt{\omega(t)}$, which we assume is not singular at any real time, we obtain the adiabatic narrow-tube formula:

$$\epsilon_A = \langle\omega\rangle_t(n+\tfrac{1}{2})\hbar + \tfrac{3}{8}s\langle\omega^{-2}\rangle_t(n+\tfrac{1}{2})^2\hbar^2 - \langle L_p\rangle_t. \tag{36}$$

By employing the single-frequency approximation to equation (27):

$$\omega(t) = \sqrt{k + 3sF^2\cos^2(t)}$$

the time averages $\langle\omega(t)\rangle_t$ and $\langle\omega^{-2}(t)\rangle_t$ appearing in (32) may further be consistently approximated in the weakly nonlinear case by the expansions

$$\langle\omega(t)\rangle_t \approx \sqrt{k}\left(1 + \frac{3}{4}\frac{s}{k}F^2 - \frac{27}{64}\left(\frac{s}{k}\right)^2 F^4\right) \tag{37}$$

and

$$\langle\omega^{-2}(t)\rangle_t \approx \frac{1}{k}\left(1 - \frac{3}{2}\frac{s}{k}F^2 + \frac{27}{8}\left(\frac{s}{k}\right)^2 F^4\right) \tag{38}$$

F being given by equation (19) in our applications; see section 5. In this approximation step we obviously need s/k to be sufficiently small. It is of interest to point out that the actual initial positions of classical trajectories on the quantized tubes do not enter explicitly in any of the formulae (38) and (39).

The geometrical-path formula (38) based on the narrow-tube approximation can now be compared numerically with formula (39), based on the adiabatic simplification of (38). As reference results we have also performed calculations of the related quasi-angle $\theta = \epsilon T/\hbar \pmod{2\pi}$ based on the semiclassical numerical routines developed in [4].

5. Numerical results

In the numerical part of this work we compare the semiclassical quasi-energies ϵ and quasi-angles $\theta = \epsilon t/\hbar$ from the narrow-tube (NT) formula (35) and its adiabatic (A) limit (36) (including equations (21), (37) and (38)) with the results of the full semiclassical routines (SC) given in [4]. In tables 1(*a*)–(*d*) we kept two of the system parameters $k = 0.618^2$ and $r = 0.5$ fixed, while varying the nonlinearity parameter $s = 0.0, 0.01, 0.02, 0.03$. In a companion table, table 2, we have collected some numerical data on the centre-motion contributions $\langle -L_p\rangle_{NT}$ and $\langle -L_p\rangle_A$, and the location of the periodic Milne solution $\rho_0(0)$. The tolerance 10^{-7} is used throughout the numerical computations.

In table 1(*a*) we consider the test case $s = 0$, where everything can be compared with exact results. For all nested tubes we have used the exact fixed point $q_p(0) = -r/(1-k)$ for the underlying periodic motion. Its contribution to the quasi-energy is the same for all nested tubes and given in table 2. There is complete agreement between the (NT) and (A) results, as there should be, since the analytic formulae contain no approximations for $s = 0$.

In tables 1(*b*)–(*d*) there is good agreement between the quasi-energies ϵ_{NT} and ϵ_A, although there are many approximations involved in obtaining ϵ_A. We note that the quasi-energies ϵ_{NT} are larger than the corresponding ϵ_A throughout the tables. All semiclassical

Table 1. The four leading quasi-energies and the corresponding quasi angles obtained from full semiclassical (SC), narrow-tube (NT) and adiabatic narrow-tube (A) calculations. The system parameters $k = 0.618^2$ and $r = 0.5$ are fixed, while the nonlinearity parameter s is varied ((*a*) $s = 0.0$; (*b*) $s = 0.01$; (*c*) $s = 0.02$; (*d*) $s = 0.03$).

	n	ϵ_{SC}	ϵ_{NT}	ϵ_A	θ_{SC}	θ_{NT}	θ_A
$s = 0.0$	0	0.4102	0.410 120	0.410 120	2.5769	2.576 862	2.576 862
	1	1.0281	1.028 120	1.028 120	0.1767	0.176 685	0.176 685
	2	1.6461	1.646 120	1.646 120	4.0597	4.059 693	4.059 693
	3	2.2641	2.264 120	2.264 120	1.6595	1.659 516	1.659 516
$s = 0.01$	0	0.4170	0.416 94	0.416 89	2.6201	2.619 70	2.619 40
	1	1.0621	1.062 11	1.062 03	0.3901	0.390 27	0.389 72
	2	1.7250	1.726 43	1.726 31	4.5555	4.564 27	4.563 51
	3	2.4047	2.409 88	2.409 73	2.5428	2.575 34	2.574 41
$s = 0.02$	0	0.4241	0.4238	0.4235	2.6642	2.6625	2.6612
	1	1.0960	1.0953	1.0950	0.6029	0.5990	0.5968
	2	1.8021	1.8042	1.8038	5.0395	5.0530	5.0502
	3	2.5401	2.5504	2.5499	3.3935	3.4581	3.4552
$s = 0.03$	0	0.4312	0.4306	0.4301	2.7096	2.7054	2.7024
	1	1.1309	1.1278	1.1271	0.8227	0.8032	0.7984
	2	1.8834	1.8796	1.8788	5.5509	5.5268	5.5214
	3	2.6930	2.6859	2.6852	4.3543	4.3097	4.3051

Table 2. Numerical data on the centre-motion contributions $\langle -L_p\rangle_{NT}$ and $\langle -L_p\rangle_A$ corresponding to the small-amplitude response, and the location of the periodic Milne solution $\rho_0(0)$. System parameters are the same as in tables 1(*a*)–(*d*).

s	$\langle -L_p\rangle_{NT}$	$\langle -L_p\rangle_A$	$\rho_0(0)$
0.0	0.101 12	0.101 12	1.272 054 63
0.01	0.101 52	0.101 50	1.269 066 32
0.02	0.101 95	0.101 84	1.266 365 15
0.03	0.102 39	0.102 14	1.263 982 77

results are in good agreement for all the quantized tubes shown in the tables. When converted to quasi-angles this agreement does not look as impressive. However, the agreement between ϵ_{NT} and ϵ_A is striking, although we did not particularly choose the system parameter k to make the adiabatic conditions ideal; the larger k is, the more adiabatic is the system.

Table 2 reveals that the centre motion does not contribute a lot to the quasi-energies in this case. The reason is mainly that the amplitude of the centre motion is small in this study. A slight increase of the centre-motion contribution as a function of the nonlinearity seems to contradict formula (21). A closer analysis shows that the nonlinearity dependence of F is responsible for this marginal increase of the centre-motion quasi-energy.

Finally in table 3 we take a brief look at the quasi-energy levels of the motion near the large, stable harmonic response for $s = 0.03$ in figure 3. We do not have a simplifying adiabatic approximation in this case but the basic narrow-tube formulae (31) and (35) could still be tested against full semiclassical calculations. For the centre motion we find a large negative contribution, $\langle -L_p\rangle_{NT} = -3.421\,940\,52$, to the quasi energy. The quasi-energy 'well' seems to be much deeper in this part of phase space. The periodic Milne solution is found at $\rho_0(0) = 0.404\,568\,064$. To our surprise the leading-order formula (31) is

Table 3. Quasi-energy levels and the corresponding quasi-angle associated with the large, stable harmonic response, (see figures 2 and 3). Indices refer to full semiclassical (sc) and leading-order (o) narrow-tube calculations. The system parameters are the same as in table 1(d).

n	ϵ_{SC}	ϵ_O	θ_{SC}	θ_O
0	−2.8445	−2.8439	−5.3059	−5.3024
1	−1.6907	−1.6878	−4.3399	−4.3217
2	−0.5391	−0.5317	−3.3871	−3.3410
3	0.6096	0.6243	3.8300	3.9229

optimal, the correction term in (35) causes a positive shift (of the order 0.012 for $n = 0$) in the quasi-energy, further away from the full semiclassical result. We realize that the approximate averaging procedure for estimating the contribution from the cubic and quartic terms in the Hamiltonian (32), and which led us to neglect the cubic term, may not be appropriate if the periodic response q_p has a large amplitude. The negative quasi-angles in table 3 reflect the negative values of the corresponding quasi-energies.

6. Discussion

We have considered (principle) quasi-energy states that are localized on vortex tubes (tori) around a stable periodic orbit in phase space. For such states the narrow-tube formulae seem to be valuable semiclassical estimates of the quantal quasi-energies—as a direct comparison with the full semiclassical quantization procedure shows [4].

Furthermore, the quasi-energy has been decomposed into two contributions, namely a contribution from the centre motion which is common to all vortex tubes around the periodic orbit in question, and individual terms which depend on the quantum number n of the individual quantized flux tubes.

Sometimes good analytic descriptions are available for the centre motion, as for the small-amplitude periodic orbit of present Duffing oscillator. However, in cases where no analytic solutions are available the narrow-tube formulae are easy to use numerically. The location of the periodic centre motion of the nonlinear system is found by a standard Newton routine, making use of the linearized variational equation for the system. With the localization of the centre one can also include the calculation of $\langle L_p(t)\rangle_t$. In addition, certain real and periodic solutions of an auxiliary Milne equation play an essential role in the narrow-tube formula. In a separate program the nonlinear system equations are calculated again, now with the correct initial condition for the periodic response, including also the corresponding Milne equation for which the periodic solution can be found by a Newton algorithm, as before. Good initial conditions are provided by the adiabatic periodic Milne solution in our calculations. One can also find the quantities $\langle \rho^{-2}(t)\rangle_t$ and $\langle \rho^4(t)\rangle_t$ by integrating over a single period in the last Newton iteration.

We also would like to draw attention to a completely different approach to semiclassical eigenvalue quantization based on Gutzwiller's periodic orbit theory, which has been applied almost exclusively to chaotic, *time-independent* Hamiltonian systems (with two or more degrees of freedom) allowing only unstable periodic motion. However, in a remarkable study of *time-independent* Hamiltonian systems, Miller [11] eliminates some of the principle shortcomings of Gutzwiller's original theory for isolated elliptic islands and arrives at an interesting formula expressing the quantized energies as the sum of two classes of contributions: one is associated to the stable periodic orbit itself and the other contribution is generated by deviating, harmonic (normal mode) vibrations about the centre motion.

This result of Miller has a striking similarity to our result for a *time-periodic* Hamiltonian system. In the leading-order narrow-tube limit our quasi-energy formula (31), with the trivial Brillouin-zone contribution included [12], is given by

$$\epsilon_0^{(m,n)} = m\omega\hbar - \langle L_p(t)\rangle_t + (n + \tfrac{1}{2})\langle \rho^{-2}(t)\rangle_t \hbar \qquad m = 0, \pm 1, \pm 2, \ldots \qquad n = 0, 1, 2, \ldots$$

where $\omega = 2\pi/T$ is the angular frequency of the periodic response induced by the external forcing; The first two terms of this formula give the quasi-energy contribution from the time dependence of the periodic orbit. The differences compared to Miller's formula (2.24) originate from a periodicity requirement in *time* rather than confinement (and quantization) in *space* along the trajectory. The deviating, parametrically driven, harmonic vibrations about the centre motion result in the contribution of the third term in the above formula. The harmonic oscillator deviations of Miller are replaced by parametrically driven harmonic oscillator deviations in our case, with $\langle \rho^{-2}(t)\rangle_t$ being the averaged angular frequency (angular winding frequency).

Further research is needed to understand the general validity of local approximations of the type presented here, and what happens, in particular, at the critical situations where the nested tubes are broken up into different types by hyperbolic fixed points (unstable periodic trajectories) of the Poincaré map.

Acknowledgment

Support by the Swedish Natural Science Research Council is gratefully acknowledged.

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